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One -Leg Hybrid Methods for Solving ODEs and DAEs

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ABSTRACT

This paper introduces one-leg hybrid methods for solving ordinary differential equations (ODEs) and differential algebraic equations (DAEs). The order of convergence of these methods are determined and compared to the order of convergence of their twin hybrid multistep methods. The G-stability of these methods are studied. Finally, the methods are tested by solving DAEs.

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Hybrid Methods; One-leg Methods; DAEs; G-Stability.

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1 INTRODUCTION

This paper focuses on solving the initial value problems of the form:

$$f(x^{\cdot}(t); x(t); t) = 0;\ t \in [t_0 ; T] \tag{1.1a}$$

$$x(t_0) - a = 0; \tag{1.1b}$$

where $a \in R^m$ is a consistent initial value for (1.1) and the function $f : R^m \times R^m \times [t_0; T] \rightarrow R^m$ is assumed to be sufficiently smooth. If $(\partial f/\partial x^{\cdot})$ is nonsingular, then it is possible to formally solve (1.1a) for $x^{\cdot}$ in order to obtain an ordinary differential equation. However, if $(\partial f/\partial x^{\cdot})$ is singular, it is no longer possible and the solution x has to satisfy certain algebraic constraints therefore, equations (1.1) are referred to as differential algebraic equations.

Many applications of physics, engineering and circuit analysis need solutions of systems of DAEs. Some systems can be reduced to ODE system and can be solved by numerical ODE methods after reduction. Other systems in which reduction to an explicit differential systems is of the form $x^{\cdot} = f(x; t)$ are either impossible or impractical, that is because the problem is more naturally posed in the form:

$$f(t , x^{\cdot} , x , y) = 0 ; \tag{1.2a}$$

$$G(t , x , y) = 0 ; \tag{1.2b}$$

and a reduction might reduce the sparseness of Jacobian matrices. These systems are then solved directly, [17].

M. Ebadi and M.Y. Gokhale [9,10,11] have presented class 2+1 hybrid BDF-like methods, hybrid BDF methods (HBDF), and new hybrid methods for the numerical solution of IVPs. This provides the methods with wide stability regions and good performance in solving CPU time compared to the extended BDF (EBDF) and modified extended BDF (MEBDF) methods,[3].

The author presents two classes of hybrid methods which have better stability regions, [15].

The first hybrid class takes the form:

$$y_{n+s} = h\ \mu\ f_n + \sum_{j=0}^{k-2} \gamma_{n-j}\ y_{n-j}\ , \tag{1.3}$$

$$y_n + \sum_{j=1}^{k} \alpha_{n-j}\ y_{n-j} = h\ \beta_s\ (f_{n+s} - \beta^* f_{n-1}), \tag{1.4}$$

where $f_{n+s} = f(t_{n+s}; y_{n+s})$; $t_{n+s} = t_n + sh$; $-1 < s < 1$ and β_s, α_{n-j} , j = 1; 2, … , k; are parameters to be determined as functions of s and β^* .The method with step k has order p = k and y_{n+s} has order k – 1. To evaluate the value of y_{n+s} at off-step point, i.e. t_{n+s}, consider the nodes t_n (double node), t_{n-1} , … , t_{n-k} (simple nodes). For these data points the following scheme of divided differences has been used:

$$\begin{array}{cccccc}
t_n & y_n & & & & \\
 & & y'_n & & & \\
t_n & y_n & & \frac{hy'_n-\nabla y_n}{h^2} & & \\
 & & \frac{\nabla y_n}{h} & & \frac{hy'_n-\nabla y_n-\frac{1}{2}\nabla^2 y_n}{2!h^3} & \\
t_{n-1} & y_{n-1} & & \frac{\nabla^2 y_n}{2!h^2} & & \frac{hy'_n-\nabla y_n-\frac{1}{2}\nabla^2 y_n-\frac{\nabla^3 y_n}{3}}{3!h^4} \\
 & & \frac{\nabla y_{n-1}}{h} & & \frac{\nabla^3 y_n}{3!h^3} & \vdots \\
t_{n-2} & y_{n-2} & & \frac{\nabla^2 y_{n-1}}{2!h^2} & \vdots & \vdots \\
 & & \frac{\nabla y_{n-2}}{h} & \vdots & \vdots & \vdots \\
t_{n-3} & y_{n-3} & \vdots & \vdots & \vdots & \\
\vdots & & \vdots & \vdots & & \\
\vdots & & \vdots & & & \\
\vdots & & & & &
\end{array}$$

Applying Newton's interpolation formula for this data gives the following scheme:

$$\begin{aligned} y(t) &= y_n + (t-t_n)y'_n + (t-t_n)^2\frac{(hy'_n - \nabla y_n)}{h^2} + \\ &+(t-t_n)^2(t-t_{n-1})\frac{(hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n)}{2!h^3} + \\ &+(t-t_n)^2(t-t_{n-1})(t-t_{n-2})\frac{(hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{\nabla^3 y_n}{3})}{3!h^4} + \ldots. \end{aligned} \tag{1.5}$$

Differentiate (1.5) with respect to t

$$\begin{aligned} y'(t) &= y'_n + 2(t-t_n)\frac{(hy'_n - \nabla y_n)}{h^2} + \\ &+(2(t-t_n)(t-t_{n-1}) + (t-t_n)^2)\frac{(hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n)}{2!h^3} + \\ &+(2(t-t_n)(t-t_{n-1})(t-t_{n-2}) + (t-t_n)^2(t-t_{n-2}) + \\ &+(t-t_n)^2(t-t_{n-1}))\frac{(hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{\nabla^3 y_n}{3})}{3!h^4} + \ldots. \end{aligned} \tag{1.6}$$

Using (1.5) and (1.6) to evaluate y_{n+s} and f_{n+s} gives,

$$\begin{aligned} y(t_n + sh) &= y_n + s\ h\ f_n + s^2(h\ f_n - \nabla y_n) + \\ &+\frac{s^2(s+1)}{2!}(h\ f_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n) + \\ &+\frac{s^2(s+1)(s+2)}{3!}(h\ f_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{\nabla^3 y_n}{3}) + \ldots. \end{aligned} \tag{1.7}$$

$$\begin{aligned} f(t_{n+s}) &= f_n + 2s\frac{(hf_n - \nabla y_n)}{h} + \\ &+s(2+3s)\frac{(hf_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n)}{2!h} + \\ &+s(4+9s+4s^2)\frac{(hf_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{\nabla^3 y_n}{3})}{3!h} + \ldots. \end{aligned} \tag{1.8}$$

where f (or f(t, y)) is considered as a derivative of the solution y(t), $\nabla y_n = y_n - y_{n-1}$.

The second hybrid class takes the form:

$$y_{n+s} = h\ \mu\ f_{n+1} + \sum_{j=0}^{k-2}\gamma_{n+1-j}\ y_{n+1-j}\ , \tag{1.9}$$

$$y_{n+1} + \sum_{j=1}^{k}\alpha_{n+1-j}\ y_{n+1-j} = h\ \beta_s\ (f_{n+s} - \beta^* f_n), \tag{1.10}$$

where $0 < s < 1$ and β_s, α_{n+1-j} , j = 1, 2 , … , k; are parameters to be determined as functions of s and β^*: The method with step k has order p = k and y_{n+s} has order k – 1.

Using the above divided differences with the nodes t_{n+1} to be double node, and t_n, t_{n-1},… , t_{n-k+1} to be simple nodes and applying Newton's interpolation, $y(t_n + sh)$ takes the form:

$$y(t_n + sh) = y_{n+1} + (s-1)h\ f_{n+1} + (s-1)^2(h\ f_{n+1} - \nabla y_{n+1}) + \\ + \frac{(s-1)^2 s}{2!}(h\ f_{n+1} - \nabla y_{n+1} - \frac{1}{2}\nabla^2 y_{n+1}) + \\ + \frac{(s-1)^2 s(s+1)}{3!}(h\ f_{n+1} - \nabla y_{n+1} - \frac{1}{2}\nabla^2 y_{n+1} - \frac{1}{3}\nabla^3 y_{n+1}) + \ldots \tag{1.11}$$

One-Leg Methods

Suppose that a linear k-step method:

$$\sum_{i=0}^{k} \alpha_i\ y_{n+i} = h\ \sum_{i=0}^{k} \beta_i\ \ f(t_{n+i}, y_{n+i}), \tag{1.12}$$

is given, and that the characteristic equations:

$$\rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i \qquad \sigma(\xi) = \sum_{i=0}^{k} \beta_i \xi^i, \tag{1.13}$$

have real coefficients and no common divisor. There is also the assumption throughout the normalization that:

$$\sigma(1) = 1; \tag{1.14}$$

then the associated one-leg methods is defined by:

$$\sum_{i=0}^{k} \alpha_i\ y_{n+i} = h\ f(\ \sum_{i=0}^{k} \beta_i t_{n+i}\ , \sum_{i=0}^{k} \beta_i y_{n+i}). \tag{1.15}$$

In the one-leg methods, the derivative f is evaluated at one point only, which makes it easier to analyze. The one-leg method (1.15) may have stronger nonlinear stability properties, such as G-stability, [12,16]. On the other hand, it is known that to obtain a one-leg method of high order, the parameters α_i, β_i have to satisfy more constraints than that for linear multistep methods see,[7,8].

G-Stability

If the differential equation satisfies the one-sided Lipschitz condition:

$$\langle f(x,y) - f(x,z), y - z \rangle \le \nu \parallel y - z \parallel^2, \tag{1.16}$$

with $\nu = 0$; then the exact solutions are contractive. Consider the multistep method as a mapping $R^{n.k} \to R^{n.k}$. Let $Y_m = (y_{m+k-1}, \ldots, y_m)^T$ and consider inner product norms on $R^{n.k}$

$$\parallel Y_m \parallel_G^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} g_{ji} \langle y_{m+i-1}, y_{m+j-1} \rangle, \tag{1.17}$$

where $\langle . , . \rangle$ is the inner product on R^n used in (1.16) and k-dimensional matrix $G = (g_{ij})_{i,\ j=1, \ldots ,k}$ is assumed to be real, symmetric and positive definite.

Definition 1 [5], The one-leg method (1.15) is called G-stable, if there exists a real, symmetric and positive definite matrix G, such that for two numerical solutions $\{y_m\}$ and $\{\hat{y}_m\}$ we have:

$$\parallel Y_{m+1} - \hat{Y}_{m+1} \parallel_G \le \parallel Y_m - \hat{Y}_m \parallel_G, \tag{1.18}$$

for all step sizes h > 0 and for all differential equations satisfying (1.16) with $\nu = 0$:

Theorem 2 [2] G-stability implies A-stability.

Theorem 3 [12] Consider a method (ρ, σ). If there exists a real, symmetric and positive definite matrix G, and real numbers $a_0, \ldots, a_k$; such that:

$$\frac{1}{2}(\rho(\xi)\ \sigma(\omega) + \rho(\omega)\ \sigma(\xi)) = (\xi\omega - 1) \sum_{i,j=1}^{k} g_{ij} \xi^{i-1} \omega^{j-1} + (\sum_{i=0}^{k} a_i \xi^i)(\sum_{j=0}^{k} a_j \omega^j), \tag{1.19}$$

then the corresponding one-leg method is G-sable.

Theorem 4 [6]. If ρ and σ have no common divisor, then the method (ρ , σ) is A-stable if and only if the corresponding one-leg method is G-stable.

In sections 2, 3 the one-leg twin of the two classes of hybrid methods mentioned above with step 2 and 3 are studied respectively. Then their orders of convergence and G-stability will be investigated.

In section 4, test problems are presented. Conclusions are presented in section 5.

2 ONE- LEG METHOD FOR THE FIRST HYBRID CLASS

Here, the one-leg twin of the first class is studied when k = 2 and k = 3.

In the case of k = 2, the method takes the form:

$$\alpha_n\, y_n + \alpha_{n-1}\, y_{n-1} + \alpha_{n-2}\, y_{n-2} = h\, \beta_s\, (f_{n+s} - \beta^* f_{n-1}), \tag{2.1a}$$

$$y_{n+s} = y_n + s\, h\, f_n\, , \tag{2.1b}$$

where

$$\alpha_n = \frac{3+2s-\beta^*}{2(1-\beta^*)}\ , \quad \alpha_{n-1} = \frac{-2(1+s)}{(1-\beta^*)}\ ,$$
$$\alpha_{n-2} = \frac{-(-1-2s-\beta^*)}{2(1-\beta^*)} \text{ and } \beta_s = \frac{1}{(1-\beta^*)}.$$

Method (2.1) has order 2, its truncation error takes the form:

$$T_3 = \frac{2+3s(2+s)+\beta^*}{6(-1+\beta^*)}\, h^3\, y'''(\eta).$$

The one-leg twin of (2.1) takes the form:

$$\alpha_n\, y_n + \alpha_{n-1}\, y_{n-1} + \alpha_{n-2}\, y_{n-2} = h\ \ f(\beta_s\, t_{n+s} - \beta_s\, \beta^* t_{n-1}), \tag{2.2}$$

and has order 2 and its truncation error takes the form:

$$\bar{T}_3 = \left(\frac{1}{6} - \frac{(1+s)^2}{2(-1+\beta^*)^2}\right) h^3\, y'''(\eta).$$

If

$$s = \tfrac{1}{3}(-3 + \sqrt{3}\sqrt{1-2\,\beta^* + \beta^{*2}})$$

then, method (2.2) has order 3 and its truncation error becomes:

$$\bar{T}_4 = \frac{1-\beta^*}{36\sqrt{3}\sqrt{(1-\beta^*)^2}} h^4 y^{(4)}(\eta).$$

G- Stability Analysis

To discuss G- Stability of (2.2), using (1.8),

$$f_{n+s} = f_n;$$

Substitute f_{n+s} in equation (2.1a), it becomes:

$$\alpha_n y_n + \alpha_{n-1}\, y_{n-1} + \alpha_{n-2}\, y_{n-2} = h\, \beta_s\, (f_n - \beta^* f_{n-1}),$$

The corresponding characteristic equations are:

$$\begin{aligned} \rho(\xi) &= \alpha_n \xi^2 + \alpha_{n-1}\xi + \alpha_{n-2} \\ \sigma(\xi) &= \beta_s(\xi^2 - \beta^*\, \xi) \end{aligned}$$

Applying theorem 3, the variables a_i , i = 0, 1, 2 and g_{ij} ; i , j = 1; 2 satisfy the relations

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$$g_{11} = a_0^2,$$

$$g_{21} = \frac{-4(1+s)-4a_1 a_2(-1+\beta^*)^2+\beta^*(-3-2s+\beta^*)}{4(-1+\beta^*)^2},$$

$$g_{22} = -a_2^2 + \frac{3+2s-\beta^*}{2(-1+\beta^*)^2},$$

Choosing $\beta^*= 0.3$ and $s = -0.1$ makes $a_0 = -0.583636$,

$$a_1 = 1.54524,\ a_2 = -0.9616,\ g_{11} > 0 \text{ and } Det\left[\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\right] > 0.$$

Therefore, the matrix G is positive definite and method (2.2) is G-Stable.

In the case of k = 3 the method takes the form:

$$\alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} + \alpha_{n-2}\ y_{n-2} + \alpha_{n-3}\ y_{n-3} = h\ \beta_s\ (f_{n+s} - \beta^* f_{n-1}), \tag{2.3a}$$

$$y_{n+s} = y_n + s\ h\ f_n + s^2(h\ f_n - y_n + y_{n-1}), \tag{2.3b}$$

where

$$\alpha_n = \frac{11+12s+3s^2-2\beta^*}{6(1-\beta^*)},\ \alpha_{n-1} = \frac{-(6+10s+3s^2+\beta^*)}{2(1-\beta^*)},$$

$$\alpha_{n-2} = \frac{(3+8s+3s^2+2\beta^*)}{2(1-\beta^*)}\ ,\ \alpha_{n-3} = \frac{-(2+6s+3s^2+\beta^*)}{6(1-\beta^*)} \text{ and } \beta_s = \frac{1}{(1-\beta^*)}.$$

Method (2.3) has order 3 and its truncation error takes the form:

$$T_4 = \frac{(3+2s)(1+s(3+s))+\beta^*}{12(-1+\ \beta^*)} h^4 y^{(4)}(\eta).$$

The one-leg twin of (2.3) takes the form:

$$\alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} + \alpha_{n-2}\ y_{n-2} + \alpha_{n-3}\ y_{n-3} = h\ f(\beta_s\ t_{n+s} - \beta_s\ \beta^* t_{n-1}), \tag{2.4}$$

and has order 2, its truncation error takes the form:

$$\bar{T}_3 = -\frac{(1+s)^2\beta^*}{2(-1+\beta^*)^2} h^3 y'''(\eta).$$

G- Stability Analysis

To discuss G- Stability of (2.4), using (1.8),

$$h\ f_{n+s} =\ h\ f_n + 2s\ (h\ f_n - y_n + y_{n-1}),$$

Substitute f_{n+s} in equation (2.3a), it becomes:

$$\alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} + \alpha_{n-2}\ y_{n-2} + \alpha_{n-3}\ y_{n-3} =\ \beta_s\ ((h\ f_n + 2s\ (h\ f_n - y_n + y_{n-1})) - h\ \beta^* f_{n-1}),$$

The corresponding characteristic equations are:

$$\begin{aligned} \rho(\xi) &= (\alpha_n + 2\ s\ \ \beta_s)\xi^3 + (\alpha_{n-1} - 2s\ \beta_s)\xi^2 + \alpha_{n-2}\ \xi + \alpha_{n-3}\ , \\ \sigma(\xi) &= (1+2s)\beta_s\ \xi^3 - \beta^*\beta_s\ \xi^2. \end{aligned}$$

Applying theorem 3, the variables a_i , i = 0, 1, 2, 3 and g_{ij} ; i, j = 1, 2, 3 satisfy the relations:

$$g_{11} = a_0^2,$$

$$g_{12} = g_{21} = a_0 a_1,$$

$$g_{13} = g_{31} = \frac{12 a_0 a_2 (-1+\beta^*)^2 - \beta^* (2+3s(2+s)+\beta^*)}{12(-1+\beta^*)^2},$$

$$g_{23} = g_{32} = \frac{-3(1+2s)(6+s(14+3s)) - 12 a_2 a_3 (-1+\beta^*)^2 + \beta^*(-14-3s(10+s)+2\beta^*)}{12(-1+\beta^*)^2},$$

$$g_{22} = a_0^2 + a_1^2,$$

$$g_{33} = a_0^2 + a_1^2 + \frac{2a_2^2(-1+\beta^*)^2 - \beta^*(6+s(14+3s)+\beta^*)}{2(-1+\beta^*)^2}.$$

Choosing s = - 0.3 and β*= 0.2 makes a_0 = - 0.231455;

$$a_1 = 0.613677,\ a_2 = -0.53299,\ a_3 = 0.150767,\ g_{11} > 0,$$

$$Det\left[\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\right] > 0 \text{ and } Det\left[\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\right] > 0.$$

Therefore, the matrix G is positive definite and method (2.4) is G-Stable.

3 ONE-LEG METHOD FOR THE SECOND HYBRID CLASS

Here, the one-leg twin of the second class is studied when k = 2 and k = 3.

In the case of k = 2, the method takes the form:

$$\alpha_{n+1}\ y_{n+1} + \alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} = h\ \beta_s\ (f_{n+s} - \beta^* f_n), \qquad (3.1a)$$

$$y_{n+s} = y_{n+1} + (s-1)\ h\ f_{n+1}\ , \qquad (3.1b)$$

where

$$\alpha_{n+1} = \frac{1+2s-\beta^*}{2(1-\beta^*)},\quad \alpha_n = \frac{-2\ s}{(1-\beta^*)},$$

$$\alpha_{n-1} = \frac{-1+2s+\beta^*}{2(1-\beta^*)} \text{ and } \beta_s = \frac{1}{1-\beta^*}.$$

Method (3.1) has order 2, its truncation error takes the form:

$$T_3 = \left(\frac{1}{4} + \frac{s(1+3s)}{6(-1+\beta^*)}\right) h^3 y'''(\eta).$$

The one-leg twin of (3.1) takes the form:

$$\alpha_{n+1} y_{n+1} + \alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} = h\ \ f(\beta_s\ t_{n+s} - \beta_s\ \beta^* t_n), \qquad (3.2)$$

and has order 2, its truncation error takes the form:

$$\bar{T}_3 = \left(\frac{1}{6} - \frac{s^2}{2(-1+\beta^*)^2}\right)\ h^3\ y'''(\eta),$$

if s = - (-1+β*)/√3; then, method (3.2) has order 3 and its truncation error takes the form:

$$\bar{T}_4 = \frac{1}{36\sqrt{3}} h^4 y^{(4)}(\eta).$$

G- Stability analysis

To discuss G- Stability of (3.2), using (1.11), it's easy to show that

$$f_{n+s} = f_{n+1}$$

Substitute f_{n+s} in equation (3.1a), it becomes:

$$\alpha_{n+1}y_{n+1} + \alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} = h\ \beta_s\ (f_{n+1} - \beta^* f_n),$$

The corresponding characteristic equations are:

$$\begin{aligned} \rho(\xi) &= \alpha_{n+1}\xi^2 + \alpha_n\xi + \alpha_{n-1}\ , \\ \sigma(\xi) &= \beta_s(\ \xi^2 - \beta^*\ \xi), \end{aligned}$$

Applying theorem 3, the variables a_i, i = 0, 1, 2 and g_{ij}, i, j = 1,2 satisfy the relations:

$$g_{11} = a_0^2\ ,$$

$$g_{12} = g_{21} = \frac{4a_0a_1 4(-1+\beta^*)^2 + \beta^*(-1+2s+\beta^*)}{4(-1+\beta^*)^2},$$

$$g_{22} = -a_2^2 + \frac{1}{2(-1+\beta^*)} + \frac{s}{(-1+\beta^*)^2}$$

Choosing β^*= 0.4 and s = 0.5 makes a_0 = -1.15365;

$$a_1 = 1.39443,\ a_2 = -0.240781,\ g_{11} > 0 \text{ and } Det\left[\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\right] > 0.$$

Therefore, the matrix G is positive definite and method (3.2) is G-Stable.

In the case of k = 3 the method takes the form:

$$\alpha_{n+1}\ y_{n+1} + \alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} + \alpha_{n-2}\ y_{n-2} = h\ \beta_s\ (f_{n+s} - \beta^* f_n), \tag{3.3a}$$

$$y_{n+s} = y_{n+1} + (s-1)h\ f_{n+1} + (s-1)^2(h\ f_{n+1} - y_{n+1} + y_n), \tag{3.3b}$$

where

$$\alpha_{n+1} = \frac{2+6s+3s^2-2\beta^*}{6(1-\beta^*)},\quad \alpha_n = -\frac{(-1+4s+3s^2+\beta^*)}{2(1-\beta^*)},\quad \alpha_{n-1} = \frac{-2+2s+3s^2+2\beta^*}{2(1-\beta^*)},$$
$$\alpha_{n-2} = \frac{-(-1+3s^2+\beta^*)}{6(1-\beta^*)} \text{ and } \beta_s = \frac{1}{1-\beta^*}.$$

Method (3.3) has order 3 and its truncation error takes the form:

$$T_4 = \left(\frac{1}{9} + \frac{s(-2+s(13+8s))}{48(-1+\beta^*)}\right)h^4 y^{(4)}(\eta).$$

The one-leg method of (3.3) takes the form:

$$\alpha_{n+1}\ y_{n+1} + \alpha_n\ y_n + \alpha_{n-1}\ y_{n-1} + \alpha_{n-2}\ y_{n-2} = h\ \ f(\beta_s\ t_{n+s} - \beta_s\ \beta^* t_n), \tag{3.4}$$

and has order 2, its truncation error takes the form:

$$\bar{T}_3 = -\frac{s^2\beta^*}{2(-1+\beta^*)^2}h^3 y'''(\eta).$$

G-Stability analysis

To discuss G- Stability of (3.4), using (1.11)

$$h\ f_{n+s} = h\ f_{n+1} + 2(s-1)(h\ f_{n+1} - y_{n+1} + y_n),$$

Substitute f_{n+s} in equation (3.3a), it becomes:

$$\alpha_{n+1}\, y_{n+1} + \alpha_n\, y_n + \alpha_{n-1}\, y_{n-1} + \alpha_{n-2}\, y_{n-2} = \beta_s\, (h\, f_{n+1} + 2(s-1)(h\, f_{n+1} - y_{n+1} + y_n) - h\, \beta^*\, f_n),$$

The corresponding characteristic equations are:

$$\begin{aligned}\rho(\xi) &= (\alpha_{n+1} + 2(s-1)\beta_s\,)\,\xi^3 + (\alpha_n - 2(s-1)\beta_s\,)\,\xi^2 + \alpha_{n-1}\,\xi + \alpha_{n-2}\,,\\ \sigma(\xi) &= (2s-1)\beta_s\,\xi^3 - \beta^*\beta_s\,\xi^2).\end{aligned}$$

Applying theorem 3, the variables a_i , i = 0, 1, 2, 3 and g_{ij} , i, j = 1, 2, 3 satisfy the relations:

$$g_{11} = a_0^2,$$

$$g_{12} = g_{21} = a_0 a_1,$$

$$g_{13} = g_{31} = \frac{12a_0a_2(-1+\beta^*)^2 - \beta^*(-1+3s^2+\beta^*)}{12(-1+\beta^*)^2},$$

$$g_{23} = g_{32} = \frac{(-3(-1+2s)(-5+s(8+3s)) - 12a_2a_3(-1+\beta^*)^2 + \beta^*(13-3s(8+s)+2\beta^*)}{12(-1+\beta^*)^2},$$

$$g_{22} = a_0^2 + a_1^2,$$

$$g_{33} = a_0^2 + a_1^2 + \frac{2a_2^2(-1+\beta^*)^2 - \beta^*(-5+s(8+3s)+\beta^*)}{2(-1+\beta^*)^2}.$$

Choosing s = 0.1 and β^* = - 0.6 makes a_0 = 0.278598;

$$a_1 = -0.70395,\ a_2 = 0.572106,\ a_3 = -0.146754,,\ g_{11} > 0,$$

$$Det\left[\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\right] > 0 \text{ and } Det\left[\begin{pmatrix} g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}\right] > 0.$$

Therefore, the matrix G is positive definite and method (3.4) is G-Stable.

Remark 5 If s = 0 in the first class or s = 1 in the second class, then the parameter class [14]

$$\sum_{i=0}^{k} \alpha_i\, y_{n+i} = h\, \beta_k\, (f_{n+k} - \beta^*\, f_{n+k-1}), \tag{3.5}$$

is obtained and its one-leg twin is G-stable for k = 2:

4 NUMERICAL TESTS

Here, some numerical results are presented to evaluate the performance of the proposed technique,[4,1,13].

Test 1

Consider the differential algebraic equations:

$$y'_1(t) - t\, y'_2(t) + t^2 y'_3(t) + y_1(t) - (t+1)\, y_2(t) + (t^2 + 2t)\, y_3(t) = 0;$$

$$y'_2(t) - t\, y'_3(t) - y_2(t) + (t-1)\, y_3(t) = 0;$$

$$y_3(t) = \sin(t);$$

with the initial condition $y_1(0) = 1$; $y_2(0) = 1$; $y_3(0) = 0$;and the exact solution is $y_1(t) = \exp(-t) + t\exp(t)$; $y_2(t) = \exp(t) + t\sin(t)$; $y_3(t) = \sin(t)$:

Test 2

Consider the differential algebraic equations:

$$x'_1(t) = (\lambda - 12\text{-}t\,)x_1(t) + (2-t)\,\lambda\, z(t) + (3\text{-}t) / (2\text{-}t)\,;$$

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$$x'_2(t) = (1-\lambda) / (t-2)\, x_2(t) + (\lambda - 1)\, z(t) + 2 \exp(t);$$

$$0 = (t + 1)x_1(t) + (t^2 - 4)x_2(t) - (t^2 + t - 2) \exp(t);$$

with the initial condition $x_1(0) = 1$; $x_2(0) = 1$; $z(0) = -1/2$; and the exact solution is $x_1(t) = \exp(t)$; $x_2(t) = \exp(t)$, $z(t) = -\exp(t)/(2-t)$; where λ is a parameter and $\lambda \geq 1$.

Test 3

Consider the differential algebraic equations:

$$x'(t) = 2(1 - y) \sin(y) + x \sqrt{(1-y)} ;$$

$$0 = x^2 + (y - 1) \cos2(y);$$

with the initial condition $x(1) = 1$; $y(1) = 0$; and the exact solution is $x(t) = t \cos(1 - t^2)$, $y(t) = 1 - t^2$:

Test 4

Consider the differential algebraic equations:

$$x'(t) = f(x; t) - B(x; t)y;$$

$$0 = g(x; t);$$

Where $x'(t) = (x_1 , x_2)^T$; $f(x; t) = (1+(t-1/2) \exp(t) \; , 2t + (t^2-1/4) \exp(t))^T$; $B(x; t) = (x'_1,x'_2)^T$; $g(x; t) = ½(x^2_1 + x^2_2 - (t -1/2)^2 - (t^2 – 1/4)^2)$; with the initial condition $x_1(0) = -1/2$; $x_2(0) = -1/4$; and the exact solution is $x_1(t) = (t – 1/2)$; $x_2(t) = t^2 – 1/4$; $y(t) = \exp(t)$:

The above tests are solved by the one-leg twin of the two classes with k = 2 at different values of t. In the first method, $\beta^* = -0.4$, $s = - 0.3$ and in the second method, $\beta^* = 0.4$, $s = 0.4$: The errors of solutions of tests 1, 2, 3 and 4 are tabulated in Tables 1, 2, 3, and 4, respectively.

Table 1: The error of the first test

	t	h	$Er(y_1(t))$	$Er(y_2(t))$
First class	0.5	0.01	5.28508E-6	1.12887E-6
	1	0.01	1.20835E-5	6.13777E-6
	0.5	0.001	4.49856E-8	2.44675E-8
	1	0.001	9.01681E-8	8.65919E-8
	0.5	0.0001	4.41806E-10	2.58117E-10
	1	0.0001	8.70240E-10	8.91808E-10
Second class	0.5	0.01	3.9544E-5	7.68775E-5
	1	0.01	168652E-4	2.06118E-4
	0.5	0.001	3.39103E-7	7.74572E-7
	1	0.001	1.68448E-6	2.06791E-6
	0.5	0.0001	3.38952E-9	7.75027E-9
	1	0.0001	1.6837E-9	2.06815E-8

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Table 2: The error of the second test

	t	h	$Er(x_1(t))$	$Er(x_2(t))$	$Er(z(t))$
First class	0.5	0.01	2.98401E-6	3.01949E-6	9.71676E-6
	1	0.01	1.87239E-6	3.49243E-6	4.45479E-5
	0.5	0.001	1.99122E-8	2.77252E-8	9.12335E-8
	1	0.001	4.90379E-8	2.0546E-8	4.166861E-7
	0.5	0.0001	1.8910E-10	2.74698E-10	9.06293E-10
	1	0.0001	5.20667E-10	1.91101E-10	4.13814E-9
Second class	0.5	0.01	7.43874E-5	3.87634E-5	7.88275E-5
	1	0.01	2.52305E-4	1.32586E-5	2.87441E-4
	0.5	0.001	7.60642E-7	3.94397E-7	8.03064E-7
	1	0.001	2.55296E-6	1.34249E-6	2.91948E-6
	0.5	0.0001	7.62364E-9	3.95129E-9	8.04554E-9
	1	0.0001	2.55608E-8	1.34427E-8	2.92401E-8

Table 3: The error of the third test

	t	h	$Er(y_1(t))$	$Er(y_2(t))$
First class	1.1	0.01	8.77614E-6	1.7784E-6
	1.5	0.01	3.77352E-5	7.17592E-6
	1.1	0.01	9.58427E-8	2.30181E-8
	1.5	0.001	7.02189E-8	3.00459E-7
	1.1	0.0001	9.67034E-10	2.35144E-10
	1.5	0.0001	3.73525E-9	6.91022E-10
Second class	1.1	0.01	1.59186E-5	9.93562E-5
	1.5	0.01	4.03698E-5	4.83147E-4
	1.1	0.001	1.46259E-7	1.06048E-7
	1.5	0.001	6.14548E-6	4.56487E-7
	1.1	0.0001	1.4471E-9	1.06687E-8
	1.5	0.0001	1.15152E-8	6.00619E-8

Table 4: The error of the fourth test

	t	h	$Er(x_1(t))$	$Er(x_2(t))$	$Er(y(t))$
First class	0.1	0.01	9.24202E-8	5.30509E-8	1.12236E-6
	0.4	0.01	1.13936E-6	2.82323E-7	166416E-4
	0.1	0.001	1.31307E-9	2.37976E-9	3.96106E-8
	0.4	0.001	2.81655E-8	9.13986E-9	3.75713E-6
	0.1	0.0001	1.55239E-11	2.65254E-11	4.26026E-10
	0.4	0.0001	2.99535E-10	1.04284E-12	3.97991E-8
Second class	0.1	0.01	4.85324E-6	1.98275E-5	2.1733E-4
	0.4	0.01	1.41946E-4	211608E-4	1.92433E-2
	0.1	0.001	6.27747E-8	2.06874E-7	2.53104E-6
	0.4	0.001	1.63364E-6	3.817E-7	2.2278E-4
	0.1	0.0001	6.43586E-10	2.07603E-9	2.5685E-8
	0.4	0.0001	1.65088E-8	3.95427E-9	2.24951E-6

5 CONCLUSION

In this paper, the one leg twin of two hybrid classes presented is studied for k = 2 and k = 3. In the first class, for k = p = 2 the one-leg twin has order 2 except when $s = \frac{1}{3}\left(-3+\sqrt{3}\sqrt{(1-2\beta^* + \beta^{*2})}\right)$ it has order 3. For k = p = 3 the one-leg twin has order 2 and if $\beta^* = 0$, which leads to one leg hybrid BDF, or s = 1, which leads to the parameters class (3.5), it has order 3. In the second class, for k = p = 2 the one-leg twin has order 2 except when $s = -(-1+\beta^*) / \sqrt{3}$ it has order 3. For k = p = 3 the one-leg twin has order 2 and if $\beta^* = 0$, which leads to one leg hybrid BDF, or s = 0, which leads to the parameters class (3.5), it has order 3. The corresponding one-leg twin of the two classes is G-stable for k = 2 and k = 3. The numerical tests show that the first class gives better results than the second.

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